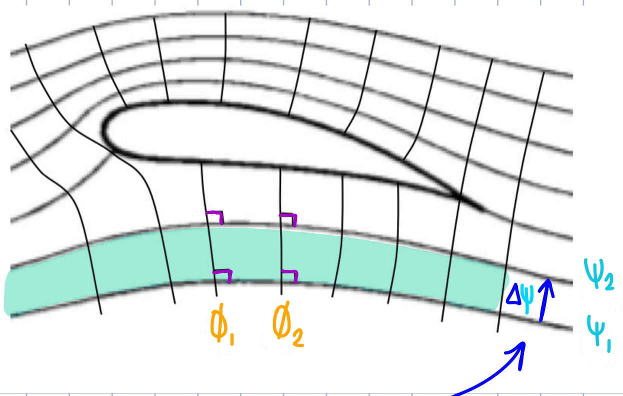
Scalar functions to describe a 2D flow, potential and stream, can be defined so long as the flow is irrotational & incompressible.



These functions are orthogonal, and relate to the velocity components by:

$$U = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$U = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

In polar coordinates:

$$V_r = \frac{\partial \phi}{\partial r} \qquad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad V_\theta = -\frac{\partial \psi}{\partial r}$$

$\Delta\psi = \dot{V} = \psi_2 - \psi_1$

Physically:

$\psi_1 = C_{ref}$ ← constant

Increment in two stream function lines = volume flow rate between those lines

$\psi_2 = C_{ref} + \Delta\psi$

$\therefore \boxed{\dot{M} = \rho \Delta \psi}$

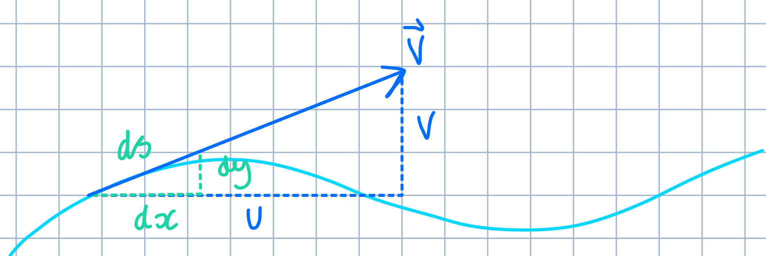
$\Delta\psi = \dot{V}$

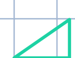
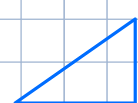
These functions both satisfy Laplace's equation, which is linear, therefore the sum of solutions is also a solution.

→ This means you can take simple flow components & add them to model more complex flow.

On streamlines, $\psi(x,y) = C$

An alternative solution is differential form, which can be integrated to give streamline equation if the stream velocity is known:



 &  Similar triangles

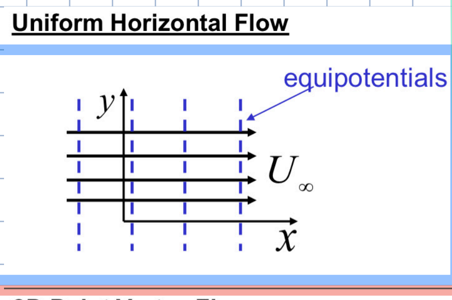
therefore $\boxed{\frac{dy}{dx} = \frac{V}{U}}$

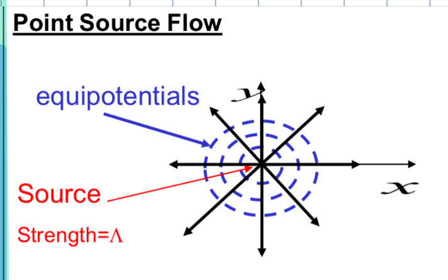
Elementary 2D Potential Solutions :

$u = U_\infty$, $v = 0$

$\phi = U_\infty x + C$

$\psi = U_\infty y + C$





$v_r = \frac{\Lambda}{2\pi r}$, $v_\theta = 0$

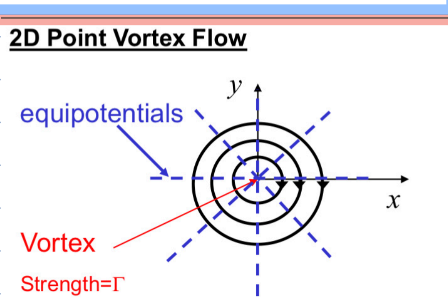
$\phi = \frac{\Lambda}{2\pi} \ln r + C$

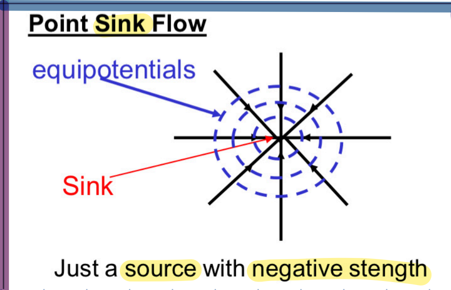
$\psi = \frac{\Lambda}{2\pi} \theta + C$

$v_r = 0$, $v_\theta = -\frac{\Gamma}{2\pi r}$

$\phi = -\frac{\Gamma}{2\pi} \theta + C$

$\psi = +\frac{\Gamma}{2\pi} \ln r + C$





$v_r = -\frac{\Lambda}{2\pi r}$, $v_\theta = 0$

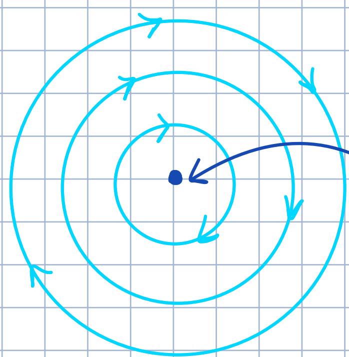
$\phi = -\frac{\Lambda}{2\pi} \ln r + C$

$\psi = -\frac{\Lambda}{2\pi} \theta + C$

Sources & sinks can model just about any inviscid flow, as long as there is no lift.

→ lift flow is modelled with vortices & doublets to provide circulation.

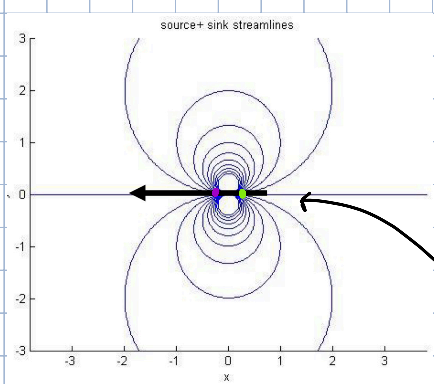
Vortex strength, $\Gamma = 0$ for irrotational flow as $\nabla \times V = 0$, and at all points in the vortex the flow is irrotational, except the singularity, where there's non-zero circularity →



$\Gamma = 0$ - flow moving around but it isn't rotating

at singularity $\Gamma \neq 0$

Doublet : Source & sink of strength Λ (opposite strength), seperated by dist. l



$l \to 0$, but Λl = constant = κ

At this limit, the source & sink are superimposed in a doublet of finite strength κ ('kappa')

+ve direction given by arrow drawn from sink to source. A doublet has its own axes

When a doublet lies on x-axis (doublet at origin)

Doublet at Origin $\boxed{\psi = \frac{K\sin\theta}{2\pi r}}$

However often the doublet axis will be at an angle α to the x-axis, in which case:

Doublet rotated to x-axis $\boxed{\psi = \frac{K\sin(\theta-\alpha)}{2\pi r}}$

General Potential Model Comments:

- No flow can cross streamline → can be reguarded as solid boundary
- No singularities exist in real flow → singularities in potential model must not lie in flow domain

Sources, sinks, doublets & vortices placed inside solid bodies, or outside for internal flows

Boundary Conditions:

Different flow solutions are found because different geometries apply different boundary conditions to solutions of Laplace's equation:

- For the external flow over a stationary body the boundary conditions are

① far away from body ($\to \infty$) the flow vel. → uniform stream

② no flow perpendicular to solid surface

↳ surface of body = streamline

- Sources, sinks and point vortices satisfy ① if added to uniform flow
- for boundary ② there are two possibilities

Direct

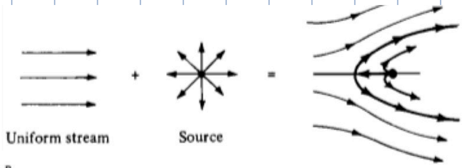
Sources, sinks & vortices added to uniform flow and any stream line of resulting flow pattern may represent body surface

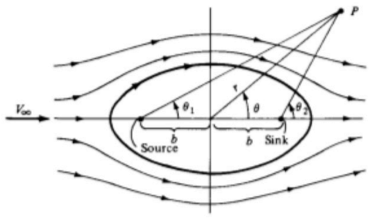
Indirect

A flow boundary specified, then sources, sinks & vortices positioned, and strength adjusted so that body surface is stream line of flow

Simple Non-Lifting Flows :

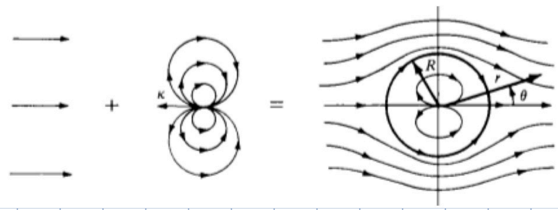
(1) uniform flow +point source
=semi-infinite body



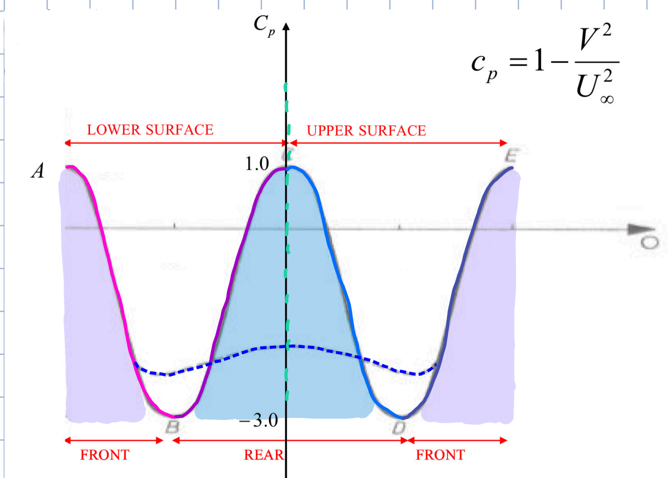


(2) uniform flow + point source and sink = axisymmetric body (eg Rankine oval)

(3) uniform flow + doublet
= flow around a cylinder



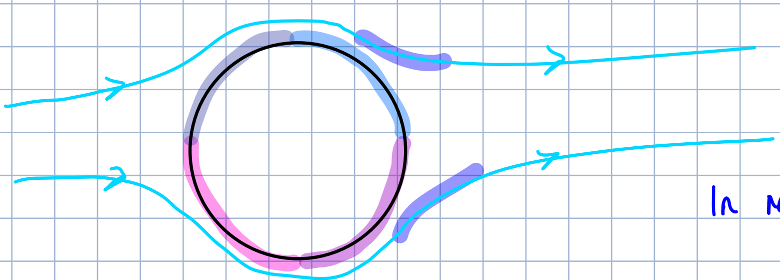
Forces on Non-Lifting Cylinder :



Pressure distribution symmetrical upper to lower surface
→ $C_L = 0$ $\qquad C_p = 1 - 4\sin^2\theta$

Pressure distribution symmetrical left to right
→ $C_D = 0$
↳ area of front = area rear

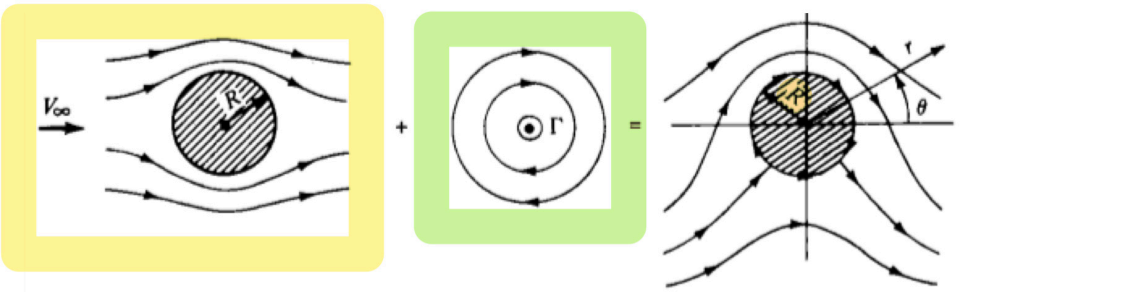
Zero drag general result for 2D potential around closed bodies
↳ D'Alembert's paradox



In reality, viscosity → seperation → $C_D \neq 0$

Lifting Cylinder Flow : add vortex to non-lifting cylinder flow

↳ streamline of vortex = circle so cylinder streamline remains



$$\psi = U_\infty r\sin\theta\left(1-\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi}\ln r-\frac{\Gamma}{2\pi}\ln R = U_\infty r\sin\theta\left(1-\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

Freestream and doublet · Vortex · Arbitrary constant

radius of cylinder

Arbitrary constant → added to match streamfunction value at surface to non-lifting case
→ when differentiated, constant removed so doesn't change velocity pattern

Differentiating gives velocity terms:

$$\boxed{V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left(1-\frac{R^2}{r^2}\right)U_\infty\cos\theta}\ , \quad \boxed{V_\theta = -\frac{\partial\psi}{\partial r} = -\left(1+\frac{R^2}{r^2}\right)U_\infty\sin\theta - \frac{\Gamma}{2\pi r}}$$

On cylinder surface where $R=r$: $V_r = 0$, $V_\theta = \underbrace{-2U_\infty\sin\theta}_{\text{freestream \& doublet}} - \underbrace{\frac{\Gamma}{2\pi R}}_{\text{Vortex}}$

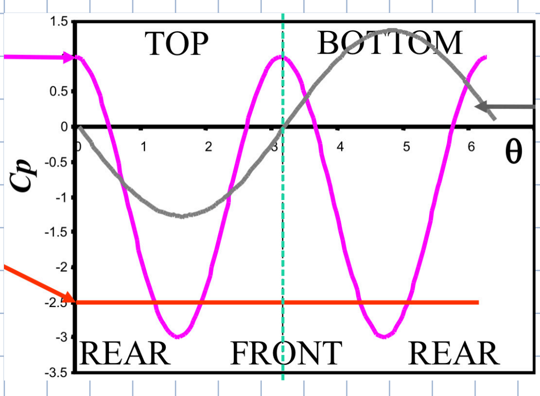
Surface Pressure found with

$$C_p = 1-\frac{V^2}{U_\infty^2} = 1-\frac{V_\theta^2}{U_\infty^2}$$

$V^2 \rightarrow V_\theta^2$ as $V^2 = \sqrt{V_\theta^2+V_r^2}$ and $V_r = 0$ at surface

$$\boxed{C_p = 1-4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi R U_\infty} + \left(\frac{\Gamma}{2\pi R U_\infty}\right)^2}$$

- basic cylinder flow
- Asymetric swirl comp. due to vortex - freestream interaction
- constant component (lift & drag contribution zero)



- Lift contribution ≠ 0 due to vortex + freestream
- +ve lift from suction on upper surface and positive pressure on lower surface
  → no top to bottom symmetry

- Drag = 0 because front & rear pressure distributions are equal

Lift Generation:

$$\boxed{L = \rho_\infty U_\infty \Gamma}$$ Kutta-Joukowski Theorem

↳ non-dimensional: $$\boxed{C_L = \frac{L}{1/2 \rho_\infty U_\infty^2 R} = \frac{2\Gamma}{R U_\infty}}$$

2D so m instead of m^2

Stagnation Points: (cylinder)

- at a stagnation point, velocity is zero: (stagnation points not just on surface)

$$V_r = \left(1 - \frac{R^2}{r^2}\right) U_\infty \cos\theta = 0 \longrightarrow \text{to satisfy, } R = r \text{ or } \cos\theta = 0 \rightarrow \theta = \pm\frac{\pi}{2}$$

$$V_\theta = -\left(1 + \frac{R^2}{r^2}\right) U_\infty \sin\theta - \frac{\Gamma}{2\pi r} = 0 \longrightarrow \text{to satisfy } \sin\theta = \left(\frac{-\Gamma}{2\pi r U_\infty (1 + R^2/r^2)}\right)$$

subbing $R = r$

$$\sin\theta = \left(\frac{-\Gamma}{4\pi R U_\infty}\right)$$

-ve means $\sin\theta$ -ve & stagnation points ∴ in lower half plane (Γ is +ve)

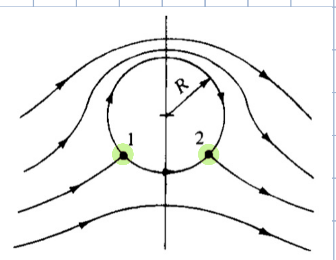
$$\theta = \sin^{-1}\left(\frac{-\Gamma}{4\pi R U_\infty}\right)$$

Possible Solutions:

- If $r = R$, stagnation points lie on cylinder surface, then

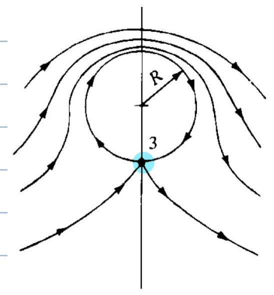
→ if $\boxed{\frac{\Gamma}{4\pi U_\infty R} < 1}$ there are two values of θ, symmetric about y-axis on surface

as $\Gamma \uparrow$, points move down cylinder



aka. $\Gamma < 4\pi U_\infty R$

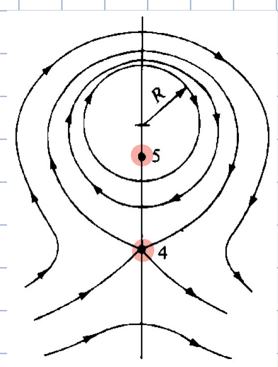
→ if $\boxed{\frac{\Gamma}{4\pi V_\infty R} = 1}$, $\sin\theta = -1$ there is one solution on y-axis on surface



aka. $\Gamma = 4\pi V_\infty R$

→ if $\boxed{\frac{\Gamma}{4\pi V_\infty R} > 1}$, we aren't restricted to cylinder surface, we can check $\cos\theta = 0$, finding solutions on y-axis

$\sin^{-1}(>1)$



aka. $\Gamma > 4\pi V_\infty R$

for $\cos\theta = 0$ $\theta = \cancel{\frac{\pi}{2}}, \frac{3\pi}{2}$

subbing into $V_\theta = -\left[V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi r}\right] = 0$

$\sin\left(\frac{\pi}{2}\right) = 1 \longrightarrow -\left[V_\infty\left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi r}\right]$

as r & R +ve, expression can never equal 0, $\therefore \theta = \frac{\pi}{2}$ invalid

for $\boxed{\theta = \frac{3\pi}{2}}$, $V_\theta = V_\infty\left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi r} = 0$ (quadratic of 'r')

$$\rightarrow V_\infty r^2 - \frac{\Gamma}{2\pi} r + R^2 V_\infty = 0$$

$$r = \frac{\frac{\Gamma}{2\pi} \pm \sqrt{\frac{\Gamma^2}{4\pi^2} - 4R^2V_\infty}}{2V_\infty} = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{\left(\frac{\Gamma}{4\pi V_\infty}\right)^2 - R^2}$$

Summary of stagnation points on cylinder:

- when $\Gamma = 0$, stagnation points lie at $\theta = 0$ and $\theta = \pi$ (on x axis)
- as Γ increased, points move down cylinder surface
- when $\Gamma = 4\pi U_\infty R$, points merge to become one at bottom of cylinder
- as Γ increased further, points split on y axis: one moves off cylinder & one inside cylinder